

# STRICHARTZ ESTIMATES FOR DIRICHLET-WAVE EQUATIONS IN TWO DIMENSIONS WITH APPLICATIONS

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**ABSTRACT.** We establish the Strauss conjecture for nontrapping obstacles when the spatial dimension  $n$  is two. As pointed out in [7] this case is more subtle than  $n = 3$  or 4 due to the fact that the arguments of the first two authors [11], Burq [1] and Metcalfe [9] showing that local Strichartz estimates for obstacles imply global ones require that the Sobolev index,  $\gamma$ , equal  $1/2$  when  $n = 2$ . We overcome this difficulty by interpolating between energy estimates ( $\gamma = 0$ ) and ones for  $\gamma = \frac{1}{2}$  that are generalizations of Minkowski space estimates of Fang and the third author [4], [5], the second author [12] and Sterbenz [14].

## 1. Introduction.

In a recent series of papers, [3], [7], techniques have been developed to prove general Strichartz estimates for wave equations outside of nontrapping obstacles. These papers relied on ideas that were used to prove the more standard  $L_t^q L_x^r$  Strichartz estimates for obstacles in [1], [9] and [11]. As was shown in [7], though, a limitation arises in the proof which is only relevant when the spatial dimension,  $n$ , equals two. This is that the  $TT^*$  arguments involving the Christ-Kiselev lemma [2] *a priori* require that the Sobolev regularity for the data in the homogeneous estimates be equal to  $\frac{1}{2}$  when  $n = 2$ , with similar restrictions on the estimates for the inhomogeneous wave equation.

As we shall see in this paper, even though we can only directly prove Strichartz estimates involving Sobolev regularity of  $\gamma = \frac{1}{2}$ , for some applications if we interpolate with trivial (energy) estimates, this is enough. In particular, we shall be able to establish the Strauss conjecture for obstacles when  $n = 2$ . Specifically, if  $\mathcal{K} \subset \mathbb{R}^2$  is a compact nontrapping obstacle with smooth boundary, then we shall be able to show that there are global small-amplitude solutions of the equation

$$(1.1) \quad \begin{cases} \square u(t, x) = F_p(u(t, x)), & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^2 \setminus \mathcal{K} \\ u(t, x) = 0, & x \in \partial\mathcal{K} \\ u|_{t=0} = f, \quad \partial_t u|_{t=0} = g \end{cases}$$

provided that

$$(1.2) \quad |F_p(u)| + |u| |F_p'(u)| \lesssim |u|^p, \quad \text{for } |u| \leq 1,$$

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and  $p$  is larger than the critical exponent for (1.1) when  $n = 2$ , which is  $p_c = (3 + \sqrt{17})/2$ , or equivalently

$$(1.3) \quad p^2 - 3p - 2 > 0, \quad \text{and } p > 0.$$

Fortunately, the critical family of estimates that we require for proving bounds for this equation involve  $\gamma = \frac{1}{2}$  (see Figure 1 below). All the other estimates, including the ones we shall use, come from interpolating between these and energy estimates.

Let us state our existence results for (1.1) with more precision. We first introduce some notation. We will denote

$$\Omega = \mathbb{R}^2 \setminus \mathcal{K}$$

and let  $\dot{H}^\gamma(\Omega)$  be the homogeneous Sobolev space of order  $\gamma$  on  $\Omega$ , with norm

$$\|f\|_{\dot{H}^\gamma(\Omega)} = \|(\sqrt{-\Delta_D})^\gamma f\|_{L^2(\Omega)},$$

with  $\Delta_D$  the Dirichlet-Laplacian in  $\Omega$ . If  $0 \leq \gamma < \frac{1}{2}$  then for  $f \in C^\infty(\Omega)$  we have

$$\|f\|_{\dot{H}^\gamma(\Omega)} \approx \|\tilde{f}\|_{\dot{H}^\gamma(\mathbb{R}^2)},$$

if  $\tilde{f}(x) = f(x)$ ,  $x \in \Omega$  and  $\tilde{f}(x) = 0$ ,  $x \in \mathcal{K}$ . Here  $\dot{H}^\gamma(\mathbb{R}^2)$  denotes the homogeneous Sobolev space with norm

$$\|g\|_{\dot{H}^\gamma(\mathbb{R}^2)}^2 = (2\pi)^{-2} \int_{\mathbb{R}^2} |\xi|^\gamma |\hat{g}(\xi)|^2 d\xi,$$

with  $\hat{g}$  denoting the Fourier transform of  $g$ . See also the introduction of [7] for a discussion of the space  $\dot{H}^\gamma(\Omega)$ .

If we also let  $\partial_j = \partial_{x_j}$ ,  $j = 1, 2$  and

$$(1.4) \quad \{Z\} = \{\partial_1, \partial_2, x_1 \partial_2 - x_2 \partial_1\},$$

then we can state our existence theorem for (1.1). The norm used in (1.5) is certainly not the best possible; see the remarks following Corollary 3.3.

**Theorem 1.1.** *Let  $n = 2$  and  $\mathcal{K}$ ,  $\Omega$  be as above. If  $p_c < p < 5$ , then there is an  $\varepsilon_0 = \varepsilon_0(p, \Omega) > 0$  such that (1.1) has a global solution satisfying*

$$Z^\alpha u(t, \cdot) \in L^{p-1}(\Omega), \quad |\alpha| \leq 1$$

*provided that the initial data  $(f, g) = (u|_{t=0}, \partial_t u|_{t=0})$  satisfies  $f|_{\partial\Omega} = 0$  and*

$$(1.5) \quad \sum_{|\alpha| \leq 2} \|Z^\alpha f\|_{L^{q_p}(\Omega)} + \sum_{|\alpha| \leq 1} \|Z^\alpha g\|_{L^{q_p}(\Omega)} < \varepsilon, \quad 0 < \varepsilon < \varepsilon_0,$$

*with  $\frac{1}{q_p} = \frac{1}{p-1} + \frac{1}{2}$ . If  $p \geq 5$ , then there is a global solution of (1.1) if (1.5) holds with  $q = q_{\tilde{p}}$ , for some  $\tilde{p} \in (p_c, 5)$ .*

Note that by Sobolev embedding

$$(1.6) \quad \sum_{|\alpha| \leq 1} (\|Z^\alpha f\|_{\dot{H}^{\gamma_p}(\Omega)} + \|Z^\alpha g\|_{\dot{H}^{\gamma_p-1}(\Omega)}) \\ \leq C_p \left( \sum_{|\alpha| \leq 2} \|Z^\alpha f\|_{L^{q_p}(\Omega)} + \sum_{|\alpha| \leq 1} \|Z^\alpha g\|_{L^{q_p}(\Omega)} \right),$$

where  $q_p$  is as above and

$$\gamma_p = 1 - \frac{2}{p-1}$$

is the scaling exponent for the equation  $\square u = |u|^p$  in two dimensions (see, e.g. [12]). In earlier works ([3], [7]) the smallness assumption on the data was based on the size of the  $\dot{H}^{\gamma_p}(\Omega) \times \dot{H}^{\gamma_p-1}(\Omega)$  norm of derivatives of  $(f, g)$  (see also (2.19) below). For technical reasons, we are led to making the somewhat stronger assumption (1.5) involving the  $L^{q_p}$ -norms, but this too is natural.

To prove Theorem 1.1 we shall require certain Strichartz estimates in  $\Omega$ . We shall postpone formulating them until they are needed in §3, but they are related to the following 2-dimensional Minkowski space estimates, which involve the angular mixed-norm spaces

$$\|f\|_{L_{|x|}^r L_\theta^2(\mathbb{R}^2)} = \left( \int_0^\infty \left( \int_0^{2\pi} |f(\rho(\cos \theta, \sin \theta))|^2 d\theta \right)^{r/2} \rho d\rho \right)^{1/r}.$$

**Proposition 1.2.** *Let  $P = \sqrt{-\Delta}$  in  $\mathbb{R}^2$ . Assume that  $(q, r) \neq (\infty, \infty)$*

$$(1.7) \quad q, r > 2 \quad \text{and} \quad \frac{1}{q} < \frac{1}{2} - \frac{1}{r},$$

or  $(q, r) = (\infty, 2)$ . Then

$$(1.8) \quad \|e^{-itP} g\|_{L_t^q L_{|x|}^r L_\theta^2(\mathbb{R} \times \mathbb{R}^2)} \leq C_{q,r} \|g\|_{\dot{H}^\gamma(\mathbb{R}^2)}, \quad \gamma = 2\left(\frac{1}{2} - \frac{1}{r}\right) - \frac{1}{q}.$$

See the following figure for the range of exponents in (1.8):

We mention that Sterbenz [14] proved related estimates where  $L_\theta^2$  is replaced by  $L_\theta^r$  (with norms of different regularity on the right). Related results are also due to Fang and Wang [4], [5] and Sogge [12] (for  $n = 3$ ). Since the proof of (1.8) is simple, we shall present it in §2 for the sake of completeness. It can be adapted to give a slightly different proof of the corresponding results in [14] and [5].

This paper is organized as follows. In the next section, we shall prove Proposition 1.2. We shall also show how it can be used to give a simple proof of Glassey's theorem [6] which says that the Strauss conjecture holds for  $\mathbb{R}_+ \times \mathbb{R}^2$ , since this will serve as a model for the more technical arguments that are needed to establish Theorem 1.1. In the final section, we shall formulate and prove the variants of (1.8) that we require and then present the proof of this theorem.

## 2. Estimates for $\mathbb{R}_+ \times \mathbb{R}^2$ and Glassey's Theorem.

We shall first prove Proposition 1.2 and then give the simple argument showing how it can be used to prove Glassey's Theorem that in  $\mathbb{R}_+ \times \mathbb{R}^2$  there is small amplitude global existence for  $\square u = |u|^p$  when  $p > p_c = (3 + \sqrt{17})/2$ .

The main step in the proof of (1.8) will be to show that

$$(2.1) \quad \|e^{-itP} f\|_{L_t^q L_{|x|}^\infty L_\theta^2(\mathbb{R} \times \mathbb{R}^2)} \leq C_q \|f\|_{L^2(\mathbb{R}^2)}, \quad \text{if } q > 2, \text{ and } \hat{f}(\xi) = 0 \text{ if } |\xi| \notin [\tfrac{1}{2}, 1].$$

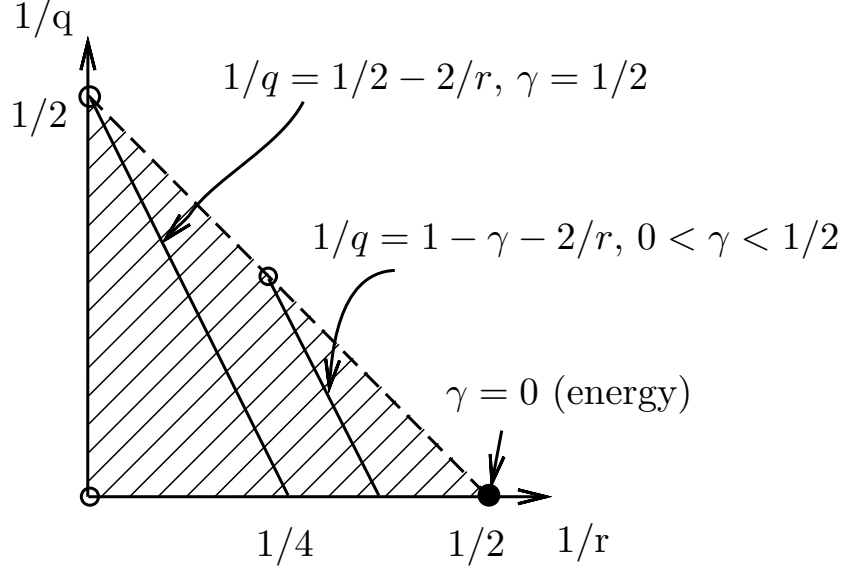


FIGURE 1. Minkowski space exponents

Since Hardy-Littlewood-Sobolev estimates give  $\dot{H}^{1-\frac{2}{r}}(\mathbb{R}^2) \subset L^r(\mathbb{R}^2)$ ,  $2 \leq r < \infty$ , we also clearly have

$$(2.2) \quad \|e^{-itP}f\|_{L_t^\infty L_{|x|}^r L_\theta^2} \leq C_r \|f\|_{\dot{H}^{1-\frac{2}{r}}},$$

since  $e^{-itP}$  is a unitary operator on  $\dot{H}^\gamma$ . The estimates (2.1) and (2.2) say that we have the estimates described in Figure 1 that respectively correspond to the (open) vertical and (half open) horizontal segments. By interpolation we conclude that, if  $q, r > 2$  and  $\frac{1}{q} < \frac{1}{2} - \frac{1}{r}$ , then

$$\|e^{-itP}f\|_{L_t^q L_{|x|}^r L_\theta^2} \leq C_{q,r} \|f\|_{L^2(\mathbb{R}^2)}, \quad \text{if } \hat{f}(\xi) = 0, \quad |\xi| \notin [\tfrac{1}{2}, 1].$$

By scaling and Littlewood-Paley theory, we obtain from this that if we remove the support assumptions on the Fourier transform, then for  $q$  and  $r$  as above, and  $(q, r) \neq (\infty, \infty)$ ,

$$(2.3) \quad \|e^{-itP}g\|_{L_t^q L_{|x|}^r L_\theta^2(\mathbb{R} \times \mathbb{R}^2)} \leq C_{q,r} \|g\|_{\dot{H}^{1-\frac{2}{r}-\frac{1}{q}}(\mathbb{R}^2)},$$

which is the inequality in Proposition 1.2.

Let us turn to the proof of (2.1). By the support assumptions for  $\hat{f}$  we have that

$$(2.4) \quad \|f\|_{L^2(\mathbb{R}^2)}^2 \approx \int_0^\infty \int_0^{2\pi} |\hat{f}(\rho(\cos \omega, \sin \omega))|^2 d\omega d\rho.$$

We expand the angular part of  $\hat{f}$  using Fourier series and find that if  $\xi = \rho(\cos \omega, \sin \omega)$ , then there are coefficients  $c_k(\rho)$  which vanish for  $\rho \notin [\frac{1}{2}, 1]$ , so that

$$\hat{f}(\xi) = \sum_k c_k(\rho) e^{ik\omega}.$$

By (2.4) and Plancherel's theorem for  $S^1$  and  $\mathbb{R}$ , we have

$$(2.5) \quad \|f\|_{L^2(\mathbb{R}^2)}^2 \approx \sum_k \int_{\mathbb{R}} |c_k(\rho)|^2 d\rho \approx \sum_k \int_{\mathbb{R}} |\hat{c}_k(s)|^2 ds,$$

where  $\hat{c}_k(s)$ ,  $s \in \mathbb{R}$ , denotes the one-dimensional Fourier transform of  $c_k(\rho)$ . Recall that (see Stein and Weiss [13] p. 137)

$$(2.6) \quad f(r(\cos \omega, \sin \omega)) = (2\pi)^{-1} \sum_k \left( i^k \int_0^\infty J_k(r\rho) c_k(\rho) \rho d\rho \right) e^{ik\omega},$$

where  $J_k$ ,  $k \in \mathbb{Z}$ , is the  $k$ -th Bessel function, defined by

$$(2.7) \quad J_k(y) = \frac{(-i)^k}{2\pi} \int_0^{2\pi} e^{iy \cos \theta - ik\theta} d\theta.$$

By (2.6) and the support properties of the  $c_k$ , if we fix  $\beta \in C_0^\infty(\mathbb{R})$  satisfying  $\beta(\tau) = 1$  for  $\tau \in [\frac{1}{2}, 1]$  and  $\beta(\tau) = 0$  for  $\tau \notin [\frac{1}{4}, 2]$ , then with  $\alpha(\rho) = \rho \beta(\rho) \in \mathcal{S}(\mathbb{R})$ , we have

$$\begin{aligned} & (e^{-itP} f)(r(\cos \omega, \sin \omega)) \\ &= (2\pi)^{-1} \sum_k \left( i^k \int_0^\infty J_k(r\rho) e^{-it\rho} c_k(\rho) \beta(\rho) \rho d\rho \right) e^{ik\omega} \\ &= (2\pi)^{-2} \sum_k \left( i^k \int_0^\infty \int_{-\infty}^\infty J_k(r\rho) e^{i\rho(s-t)} \hat{c}_k(s) \alpha(\rho) ds d\rho \right) e^{ik\omega} \\ &= (2\pi)^{-3} \sum_k \left( \int_0^\infty \int_{-\infty}^\infty \int_0^{2\pi} e^{i\rho r \cos \theta} e^{-ik\theta} e^{i\rho(s-t)} \hat{c}_k(s) \alpha(\rho) d\theta ds d\rho \right) e^{ik\omega} \\ &= (2\pi)^{-3} \sum_k \left( \int_{-\infty}^\infty \int_0^{2\pi} e^{-ik\theta} \hat{\alpha}((t-s) - r \cos \theta) \hat{c}_k(s) d\theta ds \right) e^{ik\omega}. \end{aligned}$$

As a result, we have that for any  $r \geq 0$ ,

$$(2.8) \quad \int_0^{2\pi} \left| (e^{-itP} f)(r(\cos \omega, \sin \omega)) \right|^2 d\omega = (2\pi)^{-5} \sum_k \left| \int_{-\infty}^\infty \int_0^{2\pi} e^{-ik\theta} \hat{\alpha}((t-s) - r \cos \theta) \hat{c}_k(s) d\theta ds \right|^2.$$

To estimate the right side we shall use the following.

**Lemma 2.1.** *Let  $\alpha \in \mathcal{S}(\mathbb{R})$  and  $N \in \mathbb{N}$  be fixed. Then there is a uniform constant  $C$ , which is independent of  $m \in \mathbb{R}$  and  $r \geq 0$ , so that the following inequalities hold. First,*

$$(2.9) \quad \int_0^{2\pi} |\alpha(m - r \cos \theta)| d\theta \leq C \langle m \rangle^{-N}, \quad \text{if } 0 \leq r \leq 1, \text{ or } |m| \geq 2r.$$

If  $r > 1$  and  $|m| \leq 2r$  then

$$(2.10) \quad \int_0^{2\pi} |\alpha(m - r \cos \theta)| d\theta \leq C \left( r^{-1} + r^{-\frac{1}{2}} \langle r - |m| \rangle^{-\frac{1}{2}} \right).$$

Consequently, if  $\delta > 0$ , there is a constant  $A_\delta$ , which is independent of  $t \in \mathbb{R}$  and  $r \geq 0$  so that

$$(2.11) \quad \int_{-\infty}^{\infty} \left( \int_0^{2\pi} \langle t - s \rangle^{\frac{1}{2} - \delta} |\alpha((t - s) - r \cos \theta)| d\theta \right)^2 ds \leq A_\delta.$$

If we apply (2.11) and (2.8) along with the Schwarz inequality, we conclude that if  $f$  is as in (2.1), then for  $\delta > 0$

$$\left\| e^{-itP} f \right\|_{L_{|x|}^\infty L_\theta^2}^2 \leq B_\delta \sum_k \int_{-\infty}^{\infty} |\langle t - s \rangle^{-\frac{1}{2} + \delta} \hat{c}_k(s)|^2 ds,$$

which, by Minkowski's inequality and (2.5), in turn yields (2.1).

**Proof of Lemma 2.1:** We first realize that inequalities (2.9) and (2.10) clearly imply (2.11). Also, (2.9) is trivial since  $\alpha \in \mathcal{S}$ . Therefore, we just need to prove (2.10). To do so, it suffices to show that

$$(2.12) \quad \int_0^{\pi/4} |\alpha(m - r \cos \theta)| d\theta + \int_{\pi-\pi/4}^{\pi} |\alpha(m - r \cos \theta)| d\theta \leq C r^{-\frac{1}{2}} \langle r - |m| \rangle^{-\frac{1}{2}},$$

and also

$$(2.13) \quad \int_{\pi/4}^{\pi-\pi/4} |\alpha(m - r \cos \theta)| d\theta \leq C r^{-1}.$$

In order to prove (2.12), it suffices to prove that the first integral is controlled by the right side. For if we apply this estimate to the function  $\alpha(-s)$ , we then see that the second integral satisfies the same bounds. We can estimate the first integral if we make the substitution  $u = 1 - \cos \theta$ , in which case, we see that it equals

$$\begin{aligned} \int_0^{1-1/\sqrt{2}} |\alpha((m-r) + ru)| \frac{du}{\sqrt{2u-u^2}} &\leq \int_0^{1-1/\sqrt{2}} |\alpha((m-r) + ru)| \frac{du}{\sqrt{u}} \\ &\leq C r^{-\frac{1}{2}} \int_0^{\infty} |\alpha((m-r) + u)| \frac{du}{\sqrt{u}} \\ &\leq C' r^{-\frac{1}{2}} \langle r - |m| \rangle^{-\frac{1}{2}}, \end{aligned}$$

as desired, which completes the proof of (2.12).

To prove (2.13) we just make the change of variables  $u = r \cos \theta$  and note that  $|du/d\theta| \approx r$  on the region of integration, which leads to the inequality as  $\alpha \in \mathcal{S}$ .  $\square$

We conclude this section by showing how Proposition 1.2 implies estimates that can be used to prove Glassey's [6] existence theorem for  $\square u = |u|^p$  when  $n = 2$ . Specifically, if  $u$  solves the wave equation for  $\mathbb{R} \times \mathbb{R}^2$ ,

$$(2.14) \quad \begin{cases} \square u = F \\ u|_{t=0} = f, \quad \partial_t u|_{t=0} = g, \end{cases}$$

then

$$(2.15) \quad \|u\|_{L_t^q L_{|x|}^r L_\theta^2} + \|u\|_{L_t^\infty \dot{H}^\gamma} \lesssim \|f\|_{\dot{H}^\gamma} + \|g\|_{\dot{H}^{\gamma-1}} + \|F\|_{L_t^{\tilde{q}'} L_{|x|}^{\tilde{r}'} L_\theta^2},$$

assuming that  $q, r, \tilde{q}, \tilde{r} > 2$  with  $(q, r), (\tilde{q}, \tilde{r}) \neq (\infty, \infty)$ ,  $\frac{1}{q} < \frac{1}{2} - \frac{1}{r}$ ,  $\frac{1}{\tilde{q}} < \frac{1}{2} - \frac{1}{\tilde{r}}$ , and

$$(2.16) \quad \gamma = 1 - \frac{2}{r} - \frac{1}{q}, \quad \text{and} \quad 1 - \gamma = 1 - \frac{2}{\tilde{r}} - \frac{1}{\tilde{q}}.$$

In (2.15),  $\tilde{q}'$  and  $\tilde{r}'$  denote the exponents which are conjugate to  $\tilde{q}$  and  $\tilde{r}$ , respectively, and also, here and in what follows, the space-time norms are taken over  $\mathbb{R}_+ \times \mathbb{R}^2$ . Clearly, (2.15) follows from (1.8) and energy estimates if the forcing term,  $F$ , in (2.14) vanishes. Since we are assuming (2.16) and since  $\tilde{q}' < q$ , the estimates for the inhomogeneous wave equation follow from an application of the Christ-Kiselev lemma [2] (cf. [12], pp. 136–141).

If  $\{Z\}$  are the operators in (1.4), then since they commute with  $\square$ , (2.15) implies that

$$(2.17) \quad \sum_{|\alpha| \leq 1} \left( \|Z^\alpha u\|_{L_t^q L_{|x|}^r L_\theta^2} + \|Z^\alpha u\|_{L_t^\infty \dot{H}^\gamma} \right) \lesssim \sum_{|\alpha| \leq 1} \left( \|Z^\alpha f\|_{\dot{H}^\gamma} + \|Z^\alpha g\|_{\dot{H}^{\gamma-1}} + \|Z^\alpha F\|_{L_t^{\tilde{q}'} L_{|x|}^{\tilde{r}'} L_\theta^2} \right),$$

with  $q, r, \tilde{q}', \tilde{r}'$  and  $\gamma$  as above. Let us now present the simple argument showing that this estimate implies that there are global solutions of the equation

$$(2.18) \quad \begin{cases} \square u(t, x) = F_p(u(t, x)), & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^2 \\ u|_{t=0} = f, \quad \partial_t u|_{t=0} = g, \end{cases}$$

if  $F_p$  is as in (1.2), with  $p$  as in (1.3), assuming that (when  $p_c < p < 5$ ) the initial data satisfies

$$(2.19) \quad \sum_{|\alpha| \leq 1} \left( \|Z^\alpha f\|_{\dot{H}^{\gamma_p}} + \|Z^\alpha g\|_{\dot{H}^{\gamma_p-1}} \right) < \varepsilon, \quad \gamma_p = 1 - \frac{2}{p-1},$$

with  $\varepsilon = \varepsilon(p)$  sufficiently small.

We first consider the subconformal range where  $\frac{3+\sqrt{17}}{2} = p_c < p < 5$ . This range easily lends itself to the special case of (2.17), which says that, for such  $p$ ,

$$(2.20) \quad \sum_{|\alpha| \leq 1} \left( \|Z^\alpha u\|_{L_t^{\frac{(p-1)p}{2}} L_{|x|}^p L_\theta^2} + \|Z^\alpha u\|_{L_t^\infty \dot{H}^{\gamma_p}} \right) \lesssim \sum_{|\alpha| \leq 1} \left( \|Z^\alpha f\|_{\dot{H}^{\gamma_p}} + \|Z^\alpha g\|_{\dot{H}^{\gamma_p-1}} + \|Z^\alpha F\|_{L_t^{\frac{p-1}{2}} L_{|x|}^1 L_\theta^2} \right).$$

The temporary assumption that  $p < 5$  is needed to ensure that  $(p-1)/2 < 2$ , and, therefore,  $[(p-1)/2]' > 2$ , which is the first part of the assumptions for (2.15). The more serious assumption that  $p > p_c$ , which is (1.3), is equivalent to the second part of (1.7) for the exponents on the left side of (2.20). That is, for  $p > 0$ ,

$$\frac{2}{p(p-1)} < \frac{1}{2} - \frac{1}{p} \iff p > p_c.$$

Using (2.20), we shall show that we can solve (2.18) by an iteration argument for  $p_c < p < 5$ , provided that  $\varepsilon > 0$  in (2.19) is small. To be more specific, we shall let  $u_0$  solve the Cauchy problem (2.14) with  $F \equiv 0$ . We then iteratively define  $u_k$ ,  $k \geq 1$ , by solving

$$(2.21) \quad \begin{cases} \square u_k(t, x) = F_p(u_{k-1}(t, x)), & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^2 \\ u_k|_{t=0} = f, \quad \partial_t u_k|_{t=0} = g. \end{cases}$$

Our aim is to show that if  $\varepsilon > 0$  in (2.19) is small enough, then

$$M_k = \sum_{|\alpha| \leq 1} \left( \|Z^\alpha u_k\|_{L_t^{\frac{(p-1)p}{2}} L_{|x|}^p L_\theta^2} + \|Z^\alpha u_k\|_{L_t^\infty \dot{H}^{\gamma_p}} \right)$$

must also be small.

For  $k = 0$ , it follows from (2.20) that  $M_0 \leq C_0 \varepsilon$ , with  $C_0$  a fixed constant. Clearly, (2.20) also yields that for  $k = 1, 2, 3, \dots$

$$M_k \leq C_0 \varepsilon + C_0 \sum_{|\alpha| \leq 1} \|Z^\alpha F_p(u_{k-1})\|_{L_t^{\frac{p-1}{2}} L_{|x|}^1 L_\theta^2}.$$

To control the last term, we note that our assumption (1.2) on  $F_p$  implies that

$$\sum_{|\alpha| \leq 1} |Z^\alpha F_p(v)| \lesssim |v|^{p-1} \sum_{|\alpha| \leq 1} |Z^\alpha v|, \quad \text{if } |v| \leq 1.$$

Since  $\partial_\theta = x_1 \partial_2 - x_2 \partial_1 \in \{Z\}$ , we have

$$\|v(|x| \cdot)\|_{L_\theta^\infty} \lesssim \sum_{|\alpha| \leq 1} \|Z^\alpha v(|x| \cdot)\|_{L_\theta^2},$$

and since  $0 < \gamma_p < 1$  and  $\partial_j \in \{Z\}$ ,  $j = 1, 2$ , Sobolev estimates imply that

$$\|v\|_{L^\infty(\mathbb{R}^2)} \lesssim \sum_{|\alpha| \leq 1} \|Z^\alpha v\|_{\dot{H}^{\gamma_p}(\mathbb{R}^2)}$$

so that (1.2) applies in our case. Combining the above inequalities gives

$$M_k \leq C_0 \varepsilon + C_1 C_0 M_{k-1}^p,$$

for some uniform constant  $C_1$ . Since  $M_0 \leq C_0 \varepsilon$ , we deduce from this that, if  $\varepsilon > 0$  is sufficiently small, then

$$(2.22) \quad M_k \leq 2C_0 \varepsilon, \quad k = 1, 2, 3, \dots$$

To finish the proof of the existence results for  $p_c < p < 5$ , it suffices to show that

$$A_k = \|u_k - u_{k-1}\|_{L_t^{\frac{(p-1)p}{2}} L_{|x|}^p L_\theta^2}$$

tends geometrically to zero as  $k \rightarrow \infty$ . Since  $|F_p(w) - F_p(v)| \lesssim |v - w| \cdot (|v|^{p-1} + |w|^{p-1})$  when  $|v|, |w| \leq 1$ , the proof of (2.22) can be adapted to show that, for small  $\varepsilon > 0$ , there is a uniform constant  $C$  so that

$$A_k \leq C A_{k-1} (M_{k-1} + M_{k-2})^{p-1},$$

which, by (2.22), implies that  $A_k \leq \frac{1}{2} A_{k-1}$  for small  $\varepsilon > 0$ . Since  $A_1$  is finite, the claim follows, which finishes the proof of the existence results for  $p_c < p < 5$ .



As we noted above, we cannot directly get the existence results from (2.20) if  $p \geq 5$ . However, since the assumptions (1.2) on  $F_p$  become weaker with increasing  $p$ , the above argument yields existence results for this case as well.

### 3. The Strauss conjecture for nontrapping obstacles in 2-dimensions.

The goal of this section is to show that we can solve the semilinear Dirichlet-wave equation (1.1) for small data when  $\mathcal{K} \subset \mathbb{R}^2$  is a nontrapping obstacle and, as in (1.3),  $p > p_c = \frac{3+\sqrt{17}}{2}$ . The main step will be to find a suitable variant of the Minkowski space estimate (1.8) which is valid for solutions of the linear Dirichlet-wave equation

$$(3.1) \quad \begin{cases} \square u(t, x) = F(t, x), & (t, x) \in \mathbb{R}_+ \times \Omega \\ u(t, x) = 0, & (t, x) \in \mathbb{R}_+ \times \partial\Omega \\ u|_{t=0} = f, \quad \partial_t u|_{t=0} = g, \end{cases}$$

where, as before,  $\Omega = \mathbb{R}^2 \setminus \mathcal{K}$ . As previously noted, we are in luck because the crucial estimates for (1.8) involve Sobolev regularity of  $\gamma = \frac{1}{2}$ , which is the regularity necessary for  $n = 2$  to use the techniques of [7], [1], [9] and [11], to show that local in time Strichartz estimates for  $\Omega$ , coupled with global in time estimates for  $\mathbb{R}^2$ , imply global in time estimates for  $\Omega$ . After we obtain these estimates for  $\gamma = \frac{1}{2}$ , we shall be able to obtain a family of estimates corresponding to other  $\gamma$  by interpolating with energy estimates. The range of exponents will be slightly smaller than in the previous section, in that we shall not be able to obtain indices on the open vertical line segment in Figure 1 connecting  $(\frac{1}{r}, \frac{1}{q}) = (0, 0)$  and  $(0, \frac{1}{2})$  (see Figure 2 below). Nonetheless, as we shall see, the range that we can obtain is sufficient for proving Theorem 1.1.

As in [7], due to technical difficulties in using the rotational vector fields near  $\partial\Omega$  (here  $\partial_\theta = x_1\partial_2 - x_2\partial_1$ ), we shall modify the Lebesgue spaces near  $\partial\Omega$  from those in (2.15). Specifically, given  $0 \leq \gamma < 1$ , we define

$$(3.2) \quad \|h\|_{X_{r,\gamma}} = \|h\|_{L^{s_\gamma}(|x| < 3R)} + \|h\|_{L_{|x|}^r L_\theta^2(|x| > 2R)}, \quad \text{with } \gamma = 1 - \frac{2}{s_\gamma}.$$

We fix  $R \geq 1$  large enough so that  $\mathcal{K} \subset \{|x| < R\}$ . When working with functions on  $\mathbb{R}^2$ , the norms on the right side of (3.2) are taken over  $x \in \mathbb{R}^2$  with  $|x| < 3R$  and  $|x| > 2R$  for the first and second terms, respectively. For  $\Omega$ , we define the norm in the obvious way by extending  $h$  to be equal to 0 inside  $\mathcal{K}$ .

Note that  $s_\gamma$  in (3.2) is chosen so that  $\dot{H}^\gamma(\mathbb{R}^2) \subset L^{s_\gamma}(\mathbb{R}^2)$  and  $\dot{H}^\gamma(\Omega) \subset L^{s_\gamma}(\Omega)$ , by Sobolev embedding. We conclude by Lemma 2.2 of [11] that

$$\|u\|_{L_t^2 L_x^{s_\gamma}(\mathbb{R}_+ \times \mathbb{R}^2 : |x| < 3R)} \lesssim \|f\|_{\dot{H}^\gamma(\mathbb{R}^2)} + \|g\|_{\dot{H}^{\gamma-1}(\mathbb{R}^2)}, \quad 0 < \gamma \leq \frac{1}{2}.$$

Interpolating with energy conservation lets us conclude the same bound with 2 replaced by any  $q \in [2, \infty]$ . By this and (2.15), we conclude for the Minkowski space case that, if  $u$  solves (2.14) with forcing term  $F \equiv 0$ , and  $0 < \gamma \leq \frac{1}{2}$ , then

$$(3.3) \quad \|u\|_{L_t^q X_{r,\gamma}(\mathbb{R}_+ \times \mathbb{R}^2)} + \|u\|_{L_t^\infty \dot{H}^\gamma(\mathbb{R}_+ \times \mathbb{R}^2)} + \|\partial_t u\|_{L_t^\infty \dot{H}^{\gamma-1}(\mathbb{R}_+ \times \mathbb{R}^2)} \\ \lesssim \|f\|_{\dot{H}^\gamma(\mathbb{R}^2)} + \|g\|_{\dot{H}^{\gamma-1}(\mathbb{R}^2)},$$

assuming that  $q$ ,  $r$  and  $\gamma$  are as in (2.15).

Using this estimate, the finite propagation speed for  $\square$ , and the aforementioned Sobolev inequalities, we see that we also have a local in time variant of this estimate for  $\Omega$ . Precisely, if  $u$  solves the Dirichlet-wave equation (3.1) with forcing term  $F \equiv 0$ , then

$$(3.4) \quad \|u\|_{L_t^q X_{r,\gamma}([0,1] \times \Omega)} + \|u\|_{L_t^\infty \dot{H}^\gamma([0,1] \times \Omega)} + \|\partial_t u\|_{L_t^\infty \dot{H}^{\gamma-1}([0,1] \times \Omega)} \\ \lesssim \|f\|_{\dot{H}^\gamma(\Omega)} + \|g\|_{\dot{H}^{\gamma-1}(\Omega)},$$

with the same assumptions on  $q$ ,  $r$  and  $\gamma$ .

We shall be able to use (3.3) and (3.4) to prove global variants of some of the estimates in (3.4) due to the fact that we have local energy decay estimates for the Dirichlet-wave equation (3.1). Specifically, given fixed  $R_0 > 0$  we have

$$(3.5) \quad \int_0^\infty \|u(t, \cdot)\|_{\dot{H}^1(|x| < R_0)}^2 + \|\partial_t u(t, \cdot)\|_{L^2(|x| < R_0)}^2 dt \\ \lesssim \|f\|_{H^1}^2 + \|g\|_{L^2}^2 + \int_0^\infty \|F(s, \cdot)\|_{L^2}^2 ds,$$

assuming that  $\mathcal{K}$  is nonempty and nontrapping, and that  $f(x)$ ,  $g(x)$  and  $F(t, x)$  all vanish when  $|x| > R_0$ . This was called ‘‘Hypothesis 1.1’’ in [7]. As noted there, it follows from results of Vainberg [15], but another proof can be found in Burq [1]. Also, Ralston showed in [10] that this estimate need not hold for Neumann boundary conditions in 2-dimensions, which explains why we are only treating the Dirichlet case in this paper.

Since we have (3.3)–(3.5), we can invoke Theorem 1.4 from [7] to conclude that we have global versions of (3.4) in the special case where  $\gamma = \frac{1}{2}$ . Precisely, if  $u$  solves (3.1) with  $F \equiv 0$ , then

$$(3.6) \quad \|u\|_{L_t^q X_{r,\frac{1}{2}}(\mathbb{R}_+ \times \Omega)} + \|u\|_{L_t^\infty \dot{H}^{\frac{1}{2}}(\mathbb{R}_+ \times \Omega)} + \|\partial_t u\|_{L_t^\infty \dot{H}^{-\frac{1}{2}}(\mathbb{R}_+ \times \Omega)} \\ \lesssim \|f\|_{\dot{H}^{\frac{1}{2}}(\Omega)} + \|g\|_{\dot{H}^{-\frac{1}{2}}(\Omega)},$$

assuming the following conditions on  $q$  and  $r$ ,

$$q > 2, \quad \frac{1}{2} = 1 - \frac{2}{r} - \frac{1}{q}, \quad \text{and} \quad \frac{1}{q} + \frac{1}{r} < \frac{1}{2}.$$

A limitation of Theorem 1.4 in [7] (which seems difficult to overcome) is that for  $n = 2$  it applies only to the case of  $\gamma = \frac{1}{2}$ , whereas for our existence proof we seek estimates with  $0 < \gamma < \frac{1}{2}$ . We get around this problem by an interpolation argument. Note that, by Sobolev embedding and energy conservation, if  $0 < \gamma < 1$  and  $s_\gamma$  is as in (3.2), then

$$\|u\|_{L_t^\infty L_x^{s_\gamma}(\mathbb{R}_+ \times \Omega)} + \|u\|_{L_t^\infty \dot{H}^\gamma(\mathbb{R}_+ \times \Omega)} + \|\partial_t u\|_{L_t^\infty \dot{H}^{\gamma-1}(\mathbb{R}_+ \times \Omega)} \\ \lesssim \|f\|_{\dot{H}^\gamma(\Omega)} + \|g\|_{\dot{H}^{\gamma-1}(\Omega)}.$$

Since  $s_\gamma \geq 2$ , it follows by Hölder’s inequality for  $S^1$  that the  $L^{s_\gamma}(\Omega)$  norm majorizes the  $X_{s_\gamma,\gamma}(\Omega)$  norm. Consequently, by the preceding inequality we have that

$$(3.7) \quad \|u\|_{L_t^\infty X_{s_\gamma,\gamma}(\mathbb{R}_+ \times \Omega)} + \|u\|_{L_t^\infty \dot{H}^\gamma(\mathbb{R}_+ \times \Omega)} + \|\partial_t u\|_{L_t^\infty \dot{H}^{\gamma-1}(\mathbb{R}_+ \times \Omega)} \\ \lesssim \|f\|_{\dot{H}^\gamma(\Omega)} + \|g\|_{\dot{H}^{\gamma-1}(\Omega)}, \quad 0 < \gamma < 1,$$

if  $u$  solves (3.1) with forcing term  $F \equiv 0$ .

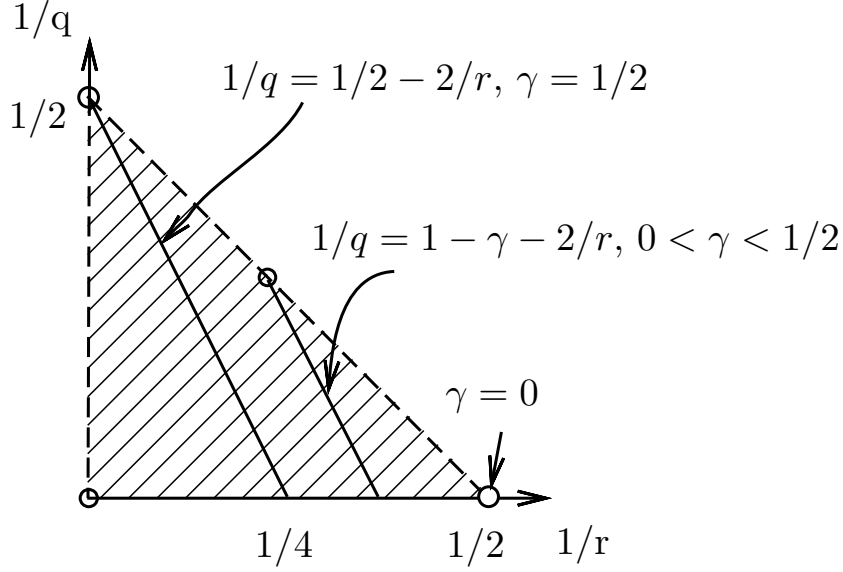


FIGURE 2. Obstacle case exponents

In Figure 2, this corresponds to the exponents on the (open) horizontal line segment corresponding to  $\frac{1}{q} = 0$ . The global estimates (3.6) correspond to the (half-open) segment where  $\frac{1}{q} = \frac{1}{2} - \frac{2}{r}$  and  $\gamma = \frac{1}{2}$  in this figure. Since the convex hull of this line segment and the horizontal segment is the shaded region in Figure 2, we conclude by interpolating between (3.6) and (3.7) that, for  $u$  solving (3.1) with vanishing forcing term, we have

$$(3.8) \quad \|u\|_{L_t^q X_{r,\gamma}(\mathbb{R}_+ \times \Omega)} + \|u\|_{L_t^\infty \dot{H}^\gamma(\mathbb{R}_+ \times \Omega)} + \|\partial_t u\|_{L_t^\infty \dot{H}^{\gamma-1}(\Omega)} \lesssim \|f\|_{\dot{H}^\gamma(\Omega)} + \|g\|_{\dot{H}^{\gamma-1}(\Omega)},$$

provided that

$$(3.9) \quad q, r > 2, \quad r < \infty, \quad \gamma = 1 - \frac{2}{r} - \frac{1}{q}, \quad \text{and} \quad \frac{1}{q} + \frac{1}{r} < \frac{1}{2}.$$

By the Christ-Kiselev lemma, if in addition  $\tilde{q}$  and  $\tilde{r}$  satisfy the variant of (3.9) corresponding to  $1 - \gamma$ ,

$$(3.10) \quad \tilde{q}, \tilde{r} > 2, \quad \tilde{r} < \infty, \quad 1 - \gamma = 1 - \frac{2}{\tilde{r}} - \frac{1}{\tilde{q}}, \quad \text{and} \quad \frac{1}{\tilde{q}} + \frac{1}{\tilde{r}} < \frac{1}{2},$$

then if  $u$  solves the linear Dirichlet-wave equation (3.1) with forcing term  $F$ , we have

$$(3.11) \quad \|u\|_{L_t^\infty \dot{H}^\gamma(\mathbb{R}_+ \times \Omega)} + \|\partial_t u\|_{L_t^\infty \dot{H}^{\gamma-1}(\mathbb{R}_+ \times \Omega)} + \|u\|_{L_t^q X_{r,\gamma}(\mathbb{R}_+ \times \Omega)} \\ \lesssim \|f\|_{\dot{H}^\gamma(\Omega)} + \|g\|_{\dot{H}^{\gamma-1}(\Omega)} + \|F\|_{L_t^{\tilde{q}'} X_{\tilde{r},1-\gamma}'(\mathbb{R}_+ \times \Omega)}.$$

Here,  $X_{\tilde{r},1-\gamma}'$  denotes the norm which is dual to that of  $X_{\tilde{r},1-\gamma}$ . For the purposes of our existence proof we do not need the exact expression for this dual norm, but use only the

following inequality. If  $h = h_1 + h_2$ , and  $h_1 = 0$  for  $|x| > 3R$ , respectively  $h_2 = 0$  for  $|x| < 2R$ , then

$$\|h\|_{X'_{\tilde{r},1-\gamma}} \leq \|h_1\|_{L^{s'_{1-\gamma}}(|x|<3R)} + \|h_2\|_{L^{\tilde{r}'}_{|x|} L^2_{\theta}(|x|>2R)},$$

where  $s'_{1-\gamma}$  and  $\tilde{r}'$  denote the exponents which are conjugate to  $s_{1-\gamma}$  and  $\tilde{r}$ , respectively. In particular, if  $\phi$  and  $\psi$  are smooth functions, with  $\phi + \psi = 1$ , and

$$\phi(x) = \begin{cases} 1, & |x| < 2R, \\ 0, & |x| > 3R, \end{cases}$$

then

$$(3.12) \quad \|h\|_{X'_{\tilde{r},1-\gamma}} \leq \|\phi h\|_{L^{s'_{1-\gamma}}(|x|<3R)} + \|\psi h\|_{L^{\tilde{r}'}_{|x|} L^2_{\theta}(|x|>2R)}.$$

As with the proof of Glassey's theorem, we need a variant of (3.11) involving the derivatives  $\{\Gamma\} = \{\partial_t, Z\}$ , where the  $\{Z\}$  vector fields are the ones in (1.4). A problem arises in establishing a version of (3.11) with derivatives, however, in that the proof of such estimates on domains with boundary, as in [7], requires local energy decay estimates that hold only for  $\gamma = \frac{1}{2}$  in dimension  $n = 2$ . Our approach will be to establish estimates with derivatives for  $\gamma = \frac{1}{2}$ , and to interpolate with (3.11) to obtain the desired estimates. For the case  $\gamma = \frac{1}{2}$ , we have the following variant of Lemma 3.3 of [7].

**Lemma 3.1.** *Suppose that  $(f, g, F)$  satisfy the Dirichlet compatibility conditions of order  $k + \frac{1}{2}$ . Then, for even integers  $k = 0, 2, 4, \dots$*

$$(3.13) \quad \sum_{|\alpha| \leq k} \left( \|\Gamma^\alpha u\|_{L_t^\infty \dot{H}^{\frac{1}{2}}(\mathbb{R}_+ \times \Omega)} + \|\Gamma^\alpha \partial_t u\|_{L_t^\infty \dot{H}^{-\frac{1}{2}}(\mathbb{R}_+ \times \Omega)} + \|\Gamma^\alpha u\|_{L_t^q X_{r,\frac{1}{2}}(\mathbb{R}_+ \times \Omega)} \right) \\ \lesssim \sum_{|\alpha| \leq k} \left( \|Z^\alpha f\|_{\dot{H}^{\frac{1}{2}}(\Omega)} + \|Z^\alpha g\|_{\dot{H}^{-\frac{1}{2}}(\Omega)} + \|\Gamma^\alpha F\|_{L_t^{\tilde{q}'} X_{\tilde{r},\frac{1}{2}}(\mathbb{R}_+ \times \Omega)} \right),$$

where  $(q, r)$  and  $(\tilde{q}, \tilde{r})$  are as in (3.9) and (3.10) for  $\gamma = \frac{1}{2}$ .

**Remark.** The condition on the data is that  $\beta f \in H_D^{k+\frac{1}{2}}$  where  $\beta$  is a compactly supported cutoff to a neighborhood of the boundary, and similarly  $\beta g \in H_D^{k-\frac{1}{2}}$ . The condition on  $F$  is that  $\beta F \in L_t^{\tilde{q}} H_D^{k-\frac{1}{2}}$ . These imply that for all  $t$ ,  $(\beta u(t, \cdot), \beta \partial_t u(t, \cdot)) \in H_D^{k+\frac{1}{2}} \times H_D^{k-\frac{1}{2}}$ , which will be used in elliptic regularity arguments. We will use the fact that, if  $f$  satisfies the  $H_D^{\frac{1}{2}}$  boundary conditions, then  $\|f\|_{\dot{H}_D^{\frac{1}{2}}(\Omega)} \approx \|f\|_{\dot{H}^{\frac{1}{2}}(\Omega)}$ , where the latter is the norm for the space of restrictions of elements in  $\dot{H}^{\frac{1}{2}}(\mathbb{R}^2)$ .

*Proof.* The proof of Lemma 3.1 follows closely the proof of Lemma 3.3 of [7], and we focus here on the modifications necessary for the above estimate. Two key estimates needed for the proof are the following. Assuming that  $F \equiv 0$  on  $|x| \geq 3R$ , then

$$(3.14) \quad \|u\|_{L_t^q X_{r,\frac{1}{2}}} + \|u\|_{L_t^\infty \dot{H}_D^{\frac{1}{2}}} + \|\partial_t u\|_{L_t^\infty \dot{H}_D^{-\frac{1}{2}}} \lesssim \|f\|_{\dot{H}_D^{\frac{1}{2}}} + \|g\|_{\dot{H}_D^{-\frac{1}{2}}} + \|F\|_{L_t^2 H_D^{-\frac{1}{2}}},$$

and, with no support assumptions on the data, the following holds

$$(3.15) \quad \|u\|_{L_t^\infty \dot{H}_D^{\frac{1}{2}}} + \|\partial_t u\|_{L_t^\infty \dot{H}_D^{-\frac{1}{2}}} + \|\beta u\|_{L_t^2 H_D^{\frac{1}{2}}} \lesssim \|f\|_{\dot{H}_D^{\frac{1}{2}}} + \|g\|_{\dot{H}_D^{-\frac{1}{2}}} + \|F\|_{L_t^{\tilde{q}'} X_{\tilde{r},\frac{1}{2}}}.$$

Estimate (3.14) follows from the estimates (2.8) of [7] and (3.11), and (3.15) follows from (3.14) by duality.

Using (3.14) and (3.15), and the case  $k = 0$  of (3.13), the argument on page 2803-2805 of [7] reduces estimate (3.13) to bounding the following quantity by the right hand side of (3.13), where  $\beta$  is a compactly supported cutoff to a neighborhood of the obstacle,

$$(3.16) \quad \sum_{j \leq k} \|\beta \partial_t^j u\|_{L_t^2 H_D^{\frac{1}{2}+k-j}} + \sum_{j \leq k+1} \|\beta \partial_t^j u\|_{L_t^\infty H_D^{\frac{1}{2}+k-j}}.$$

We first observe that the Cauchy data for  $\partial_t^j u$ ,  $j \leq k$ , belongs to  $H_D^{\frac{1}{2}} \times H_D^{-\frac{1}{2}}$ ; this is seen by using the equation to express  $(\partial_t^j u(0, \cdot), \partial_t^{j+1} u(0, \cdot))$  in terms of powers of  $\Delta$  applied to  $(f, g, F)$ , and observing that, by Sobolev embedding,

$$(3.17) \quad \begin{aligned} & \sum_{|\alpha| \leq l} \|\partial_{t,x}^\alpha F\|_{L_t^\infty \dot{H}^{\frac{1}{2}}} + \sum_{|\alpha| \leq l+1} \|\partial_{t,x}^\alpha F\|_{L_t^\infty \dot{H}^{-\frac{1}{2}}} + \sum_{|\alpha| \leq l} \|\partial_{t,x}^\alpha F\|_{L_t^2 \dot{H}^{\frac{1}{2}}} \\ & \lesssim \sum_{|\alpha| \leq l+1} \|\partial_{t,x}^\alpha F\|_{L_t^{\bar{q}} \dot{H}^{\frac{1}{2}}} + \sum_{|\alpha| \leq l+2} \|\partial_{t,x}^\alpha F\|_{L_t^{\bar{q}} \dot{H}^{-\frac{1}{2}}} \lesssim \sum_{|\alpha| \leq l+2} \|\partial_{t,x}^\alpha F\|_{L_t^{\bar{q}} X_{\frac{1}{2},r}^l}. \end{aligned}$$

By (3.15), we thus conclude that the following is bounded by the right hand side of (3.13)

$$(3.18) \quad \sum_{j \leq k} \left( \|\beta \partial_t^j u\|_{L_t^2 H_D^{\frac{1}{2}}} + \|\beta \partial_t^j u\|_{L_t^\infty H_D^{\frac{1}{2}}} + \|\beta \partial_t^{j+1} u\|_{L_t^\infty H_D^{-\frac{1}{2}}} \right).$$

To bound (3.16), it therefore suffices to bound the following quantity by the right hand side of (3.13),

$$(3.19) \quad \sum_{l \leq k/2} \left( \|\beta \partial_t^{k-2l} \Delta^l u\|_{L_t^2 H_D^{\frac{1}{2}}} + \|\beta \partial_t^{k-2l} \Delta^l u\|_{L_t^\infty H_D^{\frac{1}{2}}} + \|\beta \partial_t^{k+1-2l} \Delta^l u\|_{L_t^\infty H_D^{-\frac{1}{2}}} \right).$$

Here, we are using that we need consider only even powers of  $\partial_t$  in the first term of (3.16) since  $k$  is even, and since

$$\|\beta \partial_t^j u\|_{L_t^2 H_D^{\frac{1}{2}+k-j}}^2 \leq \|\beta \partial_t^{j+1} u\|_{L_t^2 H_D^{\frac{1}{2}+k-(j+1)}} \|\beta \partial_t^{j-1} u\|_{L_t^2 H_D^{\frac{1}{2}+k-(j-1)}}.$$

To bound (3.19), and conclude the proof, we use the equation  $(\partial_t^2 - \Delta)u = F$  to express

$$\partial_t^{k-2l} \Delta^l u = \partial_t^k u - \sum_{2j \leq k-2} \partial_t^{k-2-2j} \Delta^j F.$$

The resulting terms on the right may then be bounded in the appropriate norms using (3.18) and (3.17).  $\square$

The estimate that we shall require for the existence proof is the following. It is valid provided that  $\phi, \psi, \tilde{\phi}, \tilde{\psi} \in C^\infty(\mathbb{R}^2)$  take values in  $[0, 1]$ , with

$$(3.20) \quad \text{supp}(\phi), \text{supp}(\tilde{\phi}) \subset \{|x| < 3R\}, \quad \text{supp}(\psi), \text{supp}(\tilde{\psi}) \subset \{|x| > 2R\}, \quad \tilde{\phi} + \tilde{\psi} \geq 1.$$

We will additionally assume that each is a radial function, and that  $\phi = 1$  (respectively  $\psi = 1$ ) on a neighborhood of the support of  $\tilde{\phi}$  (respectively  $\tilde{\psi}$ ).

**Corollary 3.2.** *Suppose that  $\gamma$ ,  $(q, r)$  and  $(\tilde{q}, \tilde{r})$  are as in (3.9) and (3.10). Suppose also that  $F$  satisfies the Dirichlet compatibility conditions of order  $1 + \gamma$ . Then for the solutions  $u$  of (3.1) with vanishing Cauchy data,*

$$(3.21) \quad \sum_{|\alpha| \leq 1} \left( \|Z^\alpha u\|_{L_t^\infty L_x^{s_\gamma}} + \|\psi \Gamma^\alpha u\|_{L_t^q L_{|x|}^r L_\theta^2} + \|\phi \Gamma^\alpha u\|_{L_t^q L_x^{s_\gamma}} \right) \\ \lesssim \sum_{|\alpha| \leq 1} \left( \|\tilde{\psi} \Gamma^\alpha F\|_{L_t^{\tilde{q}} L_{|x|}^{\tilde{r}} L_\theta^2} + \|\tilde{\phi} \Gamma^\alpha F\|_{L_t^{\tilde{q}} L_x^{s'_1 - \gamma}} \right),$$

where all norms are taken over  $\mathbb{R}_+ \times \Omega$ .

*Proof.* Estimate (3.21) is obtained by interpolating estimate (3.13), which requires  $\gamma = \frac{1}{2}$  but allows arbitrarily high order powers of  $\Gamma$ , with estimate (3.11), which holds for all  $0 < \gamma < 1$ , but with 0 powers of  $\Gamma$ . We thus need to justify the interpolation step by expressing the norms in terms of analytic scales of spaces. We start by noting that

$$\sum_{|\alpha| \leq 1} \|\phi \Gamma^\alpha u\|_{L_t^q L_x^{s_\gamma}} \approx \sum_{|\alpha| \leq 1} \|\phi \partial_{t,x}^\alpha u\|_{L_t^q L_x^{s_\gamma}},$$

and

$$\sum_{|\alpha| \leq 1} \|\psi \Gamma^\alpha u\|_{L_t^q L_{|x|}^r L_\theta^2} \approx \sum_{|\alpha| \leq 1} \|\psi \partial_{t,r,\theta}^\alpha u\|_{L_t^q L_{|x|}^r L_\theta^2},$$

with similar equalities for the norms in  $F$ . By embedding  $\Omega \cap \{|x| < 3R\}$  in a compact manifold with boundary, the first norm is dominated by (with a different choice of  $\phi$ )

$$\sum_{|\alpha| \leq 1} \|D^\alpha(\phi u)\|_{L_t^q L_x^{s_\gamma}(\mathbb{R} \times \Omega')}$$

which is a Sobolev norm on a mixed-norm space. That the fractional order Sobolev norms

$$\|(1 - \partial_t^2 - \Delta_x)^{\sigma/2} u\|_{L_t^q L_x^{s_\gamma}(\mathbb{R} \times \Omega')}$$

form an analytic scale of spaces, and that norms for integer  $\sigma$  coincide with partial derivatives of order up to  $\sigma$  belonging to the mixed-norm space, follows from the fact that Calderón-Zygmund operators are bounded in mixed-norm  $L^p$  spaces, provided that all Lebesgue exponents lie in the range  $(1, \infty)$ . See Lizorkin [8] for the case of  $\mathbb{R}^n$ . The product manifold setting falls under the theory of UMD spaces; see, for example, [16].

The norms over  $\{|x| > 2R\}$  are similarly product norms over polar coordinates. Precisely,

$$\sum_{|\alpha| \leq 1} \|\psi u\|_{L_t^q L_{|x|}^r L_\theta^2} \approx \left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \left( \int_{S^1} |\psi(\rho) u(t, \rho, \theta)|^2 d\theta \right)^{r/2} \langle \rho \rangle d\rho \right)^{q/r} dt \right)^{1/q}.$$

If  $W$  denotes the forward solution operator to the wave equation on  $\Omega$ , in that  $u = WF$ , then (3.21) can be stated in terms of mapping properties of  $\phi W\tilde{\phi}$ ,  $\phi W\tilde{\psi}$ ,  $\psi W\tilde{\phi}$  and  $\psi W\tilde{\psi}$  between such spaces, where the cutoffs  $\phi$ , etc., may vary from above. For example, we need the bound, for  $k = 1$ ,

$$(3.22) \quad \sum_{|\alpha| \leq k} \|D^\alpha(\phi W\tilde{\phi} F)\|_{L_t^q L_x^{s_\gamma}(\mathbb{R} \times \Omega')} \lesssim \sum_{|\alpha| \leq k} \|D^\alpha F\|_{L_t^{\tilde{q}} L_x^{s'_1 - \gamma}(\mathbb{R} \times \Omega')},$$

where we may think of  $F$  as a function of  $(t, x) \in \mathbb{R} \times \Omega'$ . By (3.12) and Lemma 3.1, this holds for all even integers  $k$  provided  $\gamma = \frac{1}{2}$ , and by (3.11) it holds for  $k = 0$  for all  $0 < \gamma < 1$ . Since the relations (3.9) and (3.10) are linear in the reciprocals of  $s_\gamma$ ,  $q$ , and  $r$  (respectively  $s'_{1-\gamma}$ ,  $\tilde{q}$ , and  $\tilde{r}$ ), we may interpolate to obtain (3.22) for  $k = 1$  at any point in the shaded region. The estimates for the other terms follow similarly, using the embedding  $\dot{H}^\gamma \subset L^{s_\gamma}$  for the term  $\|u\|_{L_t^\infty L^{s_\gamma}}$ .

We note that if  $.2 < \gamma < .5$ , as in our application, then it suffices to consider  $k \leq 4$  for the estimate (3.13), since one may take the other endpoint with  $k = 0$  arbitrarily close to the lower right corner.  $\square$

Corollary 3.2 gives us the required estimates for the inhomogeneous equation, but we also need the following estimates for the homogeneous wave equation.

**Corollary 3.3.** *Suppose that  $\gamma$  and  $(q, r)$  are as in (3.9). Suppose also that  $f|_{\partial\Omega} = 0$ . Then for the solutions  $u$  of (3.1) with  $F \equiv 0$ ,*

$$(3.23) \quad \sum_{|\alpha| \leq 1} \left( \|Z^\alpha u\|_{L_t^\infty L_x^{s_\gamma}} + \|\psi \Gamma^\alpha u\|_{L_t^q L_{|x|}^r L_\theta^2} + \|\phi \Gamma^\alpha u\|_{L_t^q L_x^{s_\gamma}} \right) \\ \lesssim \sum_{|\alpha| \leq 2} \|Z^\alpha f\|_{L^{s'_{1-\gamma}}(\Omega)} + \sum_{|\alpha| \leq 1} \|Z^\alpha g\|_{L^{s'_{1-\gamma}}(\Omega)}.$$

This is the one step in the proof of Theorem 1.1 where condition (1.5) is used. Indeed, one can replace (1.5) by any norm condition which implies that the left hand side of (3.23) is sufficiently small (where  $\gamma = \gamma_p$ ). The norms we are using for the initial data are stronger than the norms in (2.19) using inhomogeneous Sobolev spaces, since

$$(3.24) \quad W^{1, s'_{1-\gamma}} \subset \dot{H}^\gamma, \quad L^{s'_{1-\gamma}} \subset \dot{H}^{\gamma-1}.$$

Note also that  $s'_{1-\gamma_p} = q_p$ , where  $q_p$  is as in (1.5). We use the above norm due to the difficulty in showing that  $\sum_{|\alpha| \leq k} \|Z^\alpha f\|_{\dot{H}^\gamma(\Omega)}$  defines an interpolation scale of spaces, simultaneously in  $k$  and  $\gamma$ .

Because of (3.24) (see (1.6)), we immediately find that when  $F \equiv 0$ , (3.8) and (3.13) respectively imply the somewhat weaker versions

$$(3.25) \quad \|u\|_{L_t^\infty L^{s_\gamma}} + \|u\|_{L_t^q X_{r,\gamma}} \lesssim \sum_{|\alpha| \leq 1} \|Z^\alpha f\|_{L^{s'_{1-\gamma}}} + \|g\|_{L^{s'_{1-\gamma}}},$$

and, for  $k = 0, 2, 4, \dots$  and  $(f, g)$  satisfying the compatibility conditions of order  $k + \frac{1}{2}$ ,

$$(3.26) \quad \sum_{|\alpha| \leq k} (\|\Gamma^\alpha u\|_{L_t^\infty L^4} + \|\Gamma^\alpha u\|_{L_t^q X_{r, \frac{1}{2}}}) \lesssim \sum_{|\alpha| \leq k+1} \|Z^\alpha f\|_{L^{\frac{4}{3}}} + \sum_{|\alpha| \leq k} \|Z^\alpha g\|_{L^{\frac{4}{3}}}.$$

These are the inequalities that we use in the interpolation argument to get (3.23).

The interpolation arguments are similar to those used to prove the inhomogeneous estimate (3.21). For example, for  $f$  we have

$$\sum_{|\alpha| \leq 2} \|Z^\alpha f\|_{L^{s'_{1-\gamma}}(\Omega)} \approx \sum_{|\alpha| \leq 2} \|\tilde{\psi} Z^\alpha f\|_{L^{s'_{1-\gamma}}(\Omega)} + \sum_{|\alpha| \leq 2} \|\tilde{\phi} Z^\alpha f\|_{L^{s'_{1-\gamma}}(\Omega)},$$

and

$$\begin{aligned} \sum_{|\alpha| \leq 2} \|\tilde{\psi} Z^\alpha f\|_{L^{s'_1-\gamma}(\Omega)} &\approx \sum_{|\alpha| \leq 2} \|\tilde{\psi} \partial_{r,\theta}^\alpha f\|_{L^{s'_1-\gamma}(\Omega)}, \\ \sum_{|\alpha| \leq 2} \|\tilde{\phi} Z^\alpha f\|_{L^{s'_1-\gamma}(\Omega)} &\approx \sum_{|\alpha| \leq 2} \|\tilde{\phi} \partial_x^\alpha f\|_{L^{s'_1-\gamma}(\Omega)}. \end{aligned}$$

The term  $\sum_{|\alpha| \leq 2} \|\tilde{\phi} \partial_x^\alpha f\|_{L^{s'_1-\gamma}(\Omega)}$ , can be bounded from above and below by

$$\|\tilde{\phi} f\|_{W^{2,s'_1-\gamma}} = \sum_{|\alpha| \leq 2} \|\partial_x^\alpha(\tilde{\phi} f)\|_{L^{s'_1-\gamma}}$$

(with different choices of  $\tilde{\phi}$ ), which is a standard Sobolev space norm.

If  $U$  denotes the solution operator to the wave equation (3.1) on  $\Omega$ , with  $F, g \equiv 0$ , then (3.23) for this special case can be restated in terms of mapping properties of  $\phi U \tilde{\phi}$ ,  $\phi U \tilde{\psi}$ ,  $\psi U \tilde{\phi}$  and  $\psi U \tilde{\psi}$  between these spaces. For example, we need the bound, for  $k = 1$ ,

$$(3.27) \quad \sum_{|\alpha| \leq k} \|\partial_{t,x}^\alpha(\phi U \tilde{\phi} f)\|_{L_t^q L_x^{s_\gamma}(\mathbb{R} \times \Omega')} \lesssim \sum_{|\alpha| \leq k+1} \|\partial_x^\alpha f\|_{L_x^{s'_1-\gamma}(\Omega')},$$

where we may think of  $f$  as a function of  $x \in \Omega'$ . By (3.26) this holds for all even integers  $k$  provided  $\gamma = \frac{1}{2}$ , and by (3.25) it holds for  $k = 0$  for all  $0 < \gamma < 1$ . Since the relation (3.9) is linear in the reciprocals of  $s_\gamma$ ,  $q$ , and  $r$ , we may interpolate to obtain (3.27) for  $k = 1$  at any point in the shaded region of Figure 2.

Therefore, we conclude the estimate (3.23) is valid in the special case where  $g \equiv 0$ . Since similar arguments apply to the case where  $f \equiv 0$ , we get (3.23).

We shall now show how we can use (3.21) and (3.23) to prove our existence results.

**Proof of Theorem 1.1:** As in our proof of Glassey's theorem, it suffices to consider the case of  $p_c < p < 5$ . The proof in the obstacle case requires more care in selecting the indices  $q$  and  $r$ , since the case  $r = 1$  is not allowed in (3.21), as opposed to its free-space variant (2.15). We thus need to check that we can choose exponents whose ratio is  $p$ , so that we have estimates that iterate well for equations like  $\square u = |u|^p$ .

To do this, assume given  $p$  such that  $p_c < p < 5$ . We will take  $\gamma = \gamma_p$  to be the scaling index for  $\square u = |u|^p$ ,

$$\gamma_p = 1 - \frac{2}{p-1},$$

so that  $\frac{5-\sqrt{17}}{4} < \gamma_p < \frac{1}{2}$ . As noted before, the condition  $\frac{1}{2} - \gamma_p < \frac{1}{p}$  is equivalent to the condition  $p > p_c$ . This tells us that

$$(3.28) \quad \frac{1}{2} - \gamma_p < \frac{1}{r} \quad \text{if} \quad p < r < p + \delta(p),$$

for small  $\delta(p) > 0$ . We fix such an  $r$ , and determine  $\tilde{r}$  by setting  $\tilde{r}' = r/p$ . Since we may assume  $\delta(p) < p$ , then  $p < r < 2p$ , so that  $\tilde{r} \in (2, \infty)$ .



The equality in conditions (3.9) and (3.10) determines that, with  $\gamma = \gamma_p$ ,

$$q(\gamma_p, r) = \frac{p-1}{2} \cdot \frac{r}{r-(p-1)}, \quad [\tilde{q}(1-\gamma_p, \tilde{r})]' = \frac{p-1}{2} \cdot \frac{\tilde{r}'}{p\tilde{r}'-(p-1)}.$$

Since  $\tilde{r}' = r/p$ , we have

$$(3.29) \quad [\tilde{q}(1-\gamma_p, \tilde{r})]' = q(\gamma_p, r)/p.$$

The last inequalities in (3.9) and (3.10) are then equivalent to the conditions

$$\frac{1}{2} - \gamma_p < \frac{1}{r} \quad \text{and} \quad \gamma_p - \frac{1}{2} < \frac{1}{\tilde{r}}.$$

The first condition is satisfied by (3.28), and the second is satisfied since  $\gamma_p < \frac{1}{2}$ .

To conclude the verification of (3.9)-(3.10), we check that  $2 < q, r, \tilde{q}, \tilde{r} < \infty$ . By construction this holds for  $r, \tilde{r}$ . We next observe that  $q(\gamma_p, r)$  is a decreasing function of  $r$  for  $r > p$ , and

$$p+1 < q(\gamma_p, p) < 2p.$$

The first inequality here is equivalent to  $p^2 - 3p - 2 > 0$ , and the second to  $p < 5$ . Taking  $\delta(p)$  smaller if necessary, it follows that  $q(\gamma_p, r) \in (p, 2p) \subset (2, \infty)$ , and hence  $q(1-\gamma_p, \tilde{r}) \in (2, \infty)$  by (3.29).

With this choice of indices, we then have the following case of (3.21), valid for solutions  $u$  with vanishing Cauchy data:

$$(3.30) \quad \sum_{|\alpha| \leq 1} \left( \|\psi \Gamma^\alpha u\|_{L_t^q L_{|x|}^r L_\theta^2} + \|\phi \Gamma^\alpha u\|_{L_t^q L_x^{s_{\gamma_p}}} + \|Z^\alpha u_k\|_{L_t^\infty L_x^{s_{\gamma_p}}} \right) \\ \lesssim \sum_{|\alpha| \leq 1} \left( \|\tilde{\psi} \Gamma^\alpha \square u\|_{L_t^{q/p} L_{|x|}^{r/p} L_\theta^2} + \|\tilde{\phi} \Gamma^\alpha \square u\|_{L_t^{q/p} L_x^{s'_{1-\gamma_p}}} \right).$$

We now assume that the Cauchy data  $(f, g)$  satisfies the smallness condition (1.5) (where  $q_p$  is the same as our notation  $s'_{1-\gamma_p}$ ), and let  $u_0$  solve the Cauchy problem (3.1) with forcing term  $F \equiv 0$ . We then iteratively define  $u_k$ ,  $k = 1, 2, 3, \dots$ , by requiring that it solves the equation

$$\begin{cases} \square u_k(t, x) = F_p(u_{k-1}(t, x)), & (t, x) \in \mathbb{R}_+ \times \Omega \\ u_k(t, x) = 0, & (t, x) \in \mathbb{R}_+ \times \partial\Omega \\ u_k|_{t=0} = f, \quad \partial_t u_k|_{t=0} = g. \end{cases}$$

Our goal is to show that if  $\varepsilon > 0$  in (1.5) is small enough then so is

$$M_k = \sum_{|\alpha| \leq 1} \left( \|\psi \Gamma^\alpha u_k\|_{L_t^q L_{|x|}^r L_\theta^2} + \|\phi \Gamma^\alpha u_k\|_{L_t^q L_x^{s_{\gamma_p}}} + \|Z^\alpha u_k\|_{L_t^\infty L_x^{s_{\gamma_p}}} \right)$$

for every  $k = 0, 1, 2, \dots$ , where we fix  $r$  and  $q = q(\gamma_p, r)$  as in (3.21).

For  $k = 0$ , it follows from (3.23) that  $M_0 \leq C_0 \varepsilon$ , with  $C_0 > 1$  a fixed constant. For  $k = 1, 2, \dots$ , we can then use (3.21) and (3.23) to conclude that

$$(3.31) \quad \begin{aligned} M_k &\leq C_0 \varepsilon + C_1 \sum_{|\alpha| \leq 1} \left( \|\tilde{\psi} \Gamma^\alpha F_p(u_{k-1})\|_{L_t^{q/p} L_{|x|}^{r/p} L_\theta^2(|x| > 2R)} + \|\tilde{\phi} \Gamma^\alpha F_p(u_{k-1})\|_{L_t^{q/p} L_x^{s'_1 - \gamma_p}(|x| < 3R)} \right) \\ &= C_0 \varepsilon + C_1(I + II), \end{aligned}$$

with  $C_1$  another fixed constant. Assuming that  $M_{k-1} \leq 2C_0 \varepsilon$ , we will inductively show that  $M_k \leq 2C_0 \varepsilon$ .

We first note that since  $s_{\gamma_p} = p - 1 > 2$  and  $n = 2$ , it follows from Sobolev embedding on  $\Omega$  that

$$\|v\|_{L^\infty} \lesssim \sum_{|\alpha| \leq 1} \|\partial^\alpha v\|_{L^{s_{\gamma_p}}}.$$

This means that

$$\|u_{k-1}(t, x)\|_{L_t^\infty L_x^\infty} \leq CM_{k-1} \leq 2CC_0 \varepsilon \leq 1,$$

provided that  $\varepsilon$  is small enough, which verifies the condition on  $u$  in (1.2). Our assumption (1.2) on the nonlinear term,  $F_p$ , then implies that

$$(3.32) \quad \sum_{|\alpha| \leq 1} |\Gamma^\alpha F_p(u_{k-1})| \lesssim |u_{k-1}|^{p-1} \sum_{|\alpha| \leq 1} |\Gamma^\alpha u_{k-1}|.$$

Since the collection  $\{\Gamma\}$  contains  $\partial_\theta$ , by Sobolev embedding on the circle we have

$$\|v(|x| \cdot)\|_{L_\theta^\infty} \lesssim \sum_{|\alpha| \leq 1} \|\Gamma^\alpha v(|x| \cdot)\|_{L_\theta^2}, \quad |x| > 2R.$$

Consequently, since  $\Omega$  contains the set  $|x| > 2R$ , it follows for fixed  $|x| > 2R$  and  $t > 0$  that

$$\sum_{|\alpha| \leq 1} \|\Gamma^\alpha F_p(u_{k-1}(t, |x| \cdot))\|_{L_\theta^2} \lesssim \sum_{|\alpha| \leq 1} \|\Gamma^\alpha u_{k-1}(t, |x| \cdot)\|_{L_\theta^2}^p,$$

which means that  $I \leq C_2 M_{k-1}^p$ , for some uniform constant  $C_2$ .

To handle the term  $II$  in (3.31), we note that since  $s_{\gamma_p} > 2$  and  $n = 2$ , it follows from Sobolev embedding on  $\Omega \cap \{|x| < 3R\}$  that

$$\|\tilde{\phi} v\|_{L^{2(p-1)}(|x| < 3R)} \lesssim \sum_{|\alpha| \leq 1} \|\phi Z^\alpha v\|_{L^{s_{\gamma_p}}(|x| < 3R)}.$$

Since  $s'_{1-\gamma} < 2$  satisfies

$$\frac{1}{s'_{1-\gamma}} = \frac{1}{2} + \frac{1}{s_\gamma},$$

by Hölder's inequality we have, for each fixed  $t$ , that

$$\begin{aligned}
 \sum_{|\alpha| \leq 1} \|\tilde{\phi} \Gamma^\alpha F_p(u_{k-1}(t, \cdot))\|_{L^{s'_1-\gamma}(|x| < 3R)} \\
 &\lesssim \sum_{|\alpha| \leq 1} \|\tilde{\phi} u_{k-1}\|_{L^{2(p-1)}(|x| < 3R)}^{p-1} \sum_{|\alpha| \leq 1} \|\phi \Gamma^\alpha u_{k-1}(t, \cdot)\|_{L^{s_\gamma}(|x| < 3R)} \\
 &\lesssim \sum_{|\alpha| \leq 1} \|\phi \Gamma^\alpha u_{k-1}(t, \cdot)\|_{L^{s_\gamma}(|x| < 3R)}^p.
 \end{aligned}$$

This implies that we also have  $II \leq C_3 M_{k-1}^p$ , for some uniform constant  $C_3$ , which together with the bound for  $I$  gives

$$M_k \leq C_0 \varepsilon + C_1(C_2 + C_3)M_{k-1}^p \leq C_0 \varepsilon + C_1(C_2 + C_3)(2C_0 \varepsilon)^p.$$

Thus, if  $\varepsilon$  is sufficiently small, we conclude that

$$(3.33) \quad M_k \leq 2C_0 \varepsilon, \quad k = 0, 1, 2, \dots$$

To finish the proof of the existence results, for  $p_c < p < 5$  we need to show that the  $u_k$  converge to a solution of (1.1). To do this it suffices to show that

$$A_k = \|u_k - u_{k-1}\|_{L_t^\infty \dot{H}^{\gamma_p}}$$

tends geometrically to zero as  $k \rightarrow \infty$ . Since  $|F_p(v) - F_p(w)| \lesssim |v - w|(|v|^{p-1} + |w|^{p-1})$  when  $v$  and  $w$  are small, the proof of (3.33) can be adapted to show that, for small  $\varepsilon > 0$ , there is a uniform constant  $C$  so that

$$A_k \leq C A_{k-1} (M_{k-1} + M_{k-2})^{p-1},$$

which, by (3.33), implies that  $A_k \leq \frac{1}{2} A_{k-1}$  for small  $\varepsilon$ . Since  $A_1$  is finite, the claim follows, which finishes the proof of the existence results for the range of  $p_c < p < 5$ .

As in §2, the results for  $p \geq 5$  in Theorem 1.1 follow from the above and the fact that the condition (1.2) becomes weaker as  $p$  increases.  $\square$

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